ON MULTIPLICITY THEORY FOR BOOLEAN ALGEBRAS OF PROJECTIONS

BY

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ABSTRACT

The aim of this paper is to present a study of the connections between the commutants of a Boolean algebra of projections of finite multiplicity and the uniformly closed algebra generated by these projections.

Dieudonné [4] has constructed an example of a Banach space X and a Boolean algebra (B.A.) of projections $\mathfrak{B}$ of uniform multiplicity 2 such that for no choice of x_1 and x_2 in X and $0 \neq E \in \mathfrak{B}$ is EX the direct sum of the cyclic subspaces spanned by Ex_1 and Ex_2.

In this note, we shall prove that the first commutant of a B.A. of projections of finite multiplicity $\mathfrak{B}$, having Dieudonné's above mentioned property (formal definition in Section 2), consists of those spectral operators whose scalar parts belong to the algebra $\mathfrak{A}(\mathfrak{B})$ generated by $\mathfrak{B}$ in the uniform operator topology. However, we do not know if the nilpotent parts really exist.

Later, using the previous result, we shall show that if there are no nilpotent operators commuting with a B.A. of projections of finite multiplicity $\mathfrak{B}$, then its commutant is commutative, i.e, coincides with the second commutant. Using another example of Dieudonné [3] we can conclude that it must not coincide with $\mathfrak{A}(\mathfrak{B})$:

1. **Preliminaries.** For convenience we give here some definitions from Bade's papers [1] and [2]. A B.A. of projections $\mathfrak{B}$ will be called complete if for every family $\{E_\alpha\} \subseteq \mathfrak{B}$ the projections $\bigvee E_\alpha$ and $\bigwedge E_\alpha$ exist in $\mathfrak{B}$ and

$$(\bigvee E_\alpha)X = \operatorname{clm}\{E_\alpha X\}; \ (\bigwedge E_\alpha)X = \bigcap E_\alpha X$$

A B.A. of projections will be called countably decomposable if every set of disjoint projections of $\mathfrak{B}$ is at most countable.

The cyclic subspace spanned by a vector x is defined by

$$\mathfrak{M}(x) = \operatorname{clm}\{Ex \mid E \in \mathfrak{B}\}$$

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If $\mathfrak{B}$ is a complete countably decomposable B.A. of projections in a Banach space X there exists a unique multiplicity function m defined on $\mathfrak{B}$ such that $m(E)$ is the least cardinal power of a set of cyclic subspaces spanning the range of $E \in \mathfrak{B}$.

The concept of spectral operator as used here is that which was developed by Dunford in [5].

Throughout the paper X denotes a fixed Banach space, $\mathfrak{B}$ is a complete countably decomposable B.A. of projections of finite uniform multiplicity n i.e., every $\neq E \in \mathfrak{B}$ has multiplicity n, and $\mathfrak{A}(\mathfrak{B})$ is the algebra generated by $\mathfrak{B}$ in the uniform operator topology. Following [6], $\mathfrak{B}^c$ will be the commutant of $\mathfrak{B}$, i.e, the algebra of all operators commting with every $E \in \mathfrak{B}$, and $(\mathfrak{B}^c)^c$ the second commutant of $\mathfrak{B}$, i.e., the algebra of all operators which commute with every operator commuting with $\mathfrak{B}$.

Since $\mathfrak{B}$ can be regarded as the range of a spectral measure $E(\cdot)$ defined on the Borel sets of a compact Hausdorff space Ω, to every Borel measurable function f we may consider the operator (in general unbounded)

$$S(f) = \int_\Omega f(\omega)E(d\omega)$$

with the domain

$$D(S(f)) = \{x \mid x \in X;\ \lim_{m\to\infty} \int_{e_m} f(\omega)E(d\omega)x \text{ exists}\}$$

where $e_m = \{\omega \mid |f(\omega)| \leqq m\}$.

2. **Operators commuting with $\mathfrak{B}$.** In connection with Dieudonné's example [4] we give the next definition

Definition 1. We shall say that $\mathfrak{B}$ is of type (D) if for no choice $z_i \in X$; $1 \leqq i \leqq n$ and $0 \neq E \in \mathfrak{B}$

$$EX = \left[\bigvee_{i=1}^{p} \mathfrak{M}(Ez_i)\right] \oplus \left[\bigvee_{i=p+1}^{n} \mathfrak{M}(Ez_i)\right]; \qquad 1 \leqq p < n$$

Lemma 2. *Every projection commuting with $\mathfrak{B}$ belongs to it if and only if $\mathfrak{B}$ is of type (D).*

Proof. Assume $0 \neq P \in \mathfrak{B}^c$ is a projection and denote

$$F_0 = \bigvee\{E \mid E \in \mathfrak{B}, EP = 0\}$$
$$F = I - F_0$$

Since $\mathfrak{B}$ is complete $F \in \mathfrak{B}$ and obviously $PF = P$. First, we shall show that $Px = 0$ implies $Fx = 0$. Indeed, if $Px_0 = 0$ for some $0 \neq x_0 \in FX$ and

$$FX = \bigvee_{i=1}^{n} \mathfrak{M}(Fy_i)$$

then

$$x_0 = \lim_{m\to\infty} \sum_{i=1}^{n} S(f_i\chi_{e_m})Fy_i$$

where $e_m = \{\omega \mid \omega \in \Omega;\ |f_i(\omega)| \leqq m;\ i = 1,2,\cdots,n\};\ m=1,2,\cdots$. For m sufficiently large

$$0 \neq E(e_m)x_0 = \sum_{i=1}^{n} S(f_i)E(e_m)Fy_i$$

and we may suppose that $S(f_n)E(e_m)Fy_n \neq 0$. If $e \subset e_m$ is any Borel set of positive measure on which f_n satisfies the inequality $(1/m) \leqq |f_n(\omega)| \leqq m$ then

$$0 \neq E(e)Fy_n = S(f_n^{-1})E(e)Fx_0 - \sum_{i=1}^{n-1} S(f_if_n^{-1})E(e)Fy_i$$

and, therefore

$$E(e)FX = \mathfrak{M}(E(e)Fx_0) \vee \mathfrak{M}(E(e)Fy_1) \vee \cdots \vee \mathfrak{M}(E(e)Fy_{n-1})$$

In conclusion, there exist systems $\{0 \neq G \in \mathfrak{B}; G \leqq F; x_0, x_1, \cdots, x_{n-1}\}$ such that

$$GX = \bigvee_{i=0}^{n-1} \mathfrak{M}(Gx_i).$$

For each such system we can assume that $x_1, \cdots, x_{n-1}$ were arranged such that

$$PGx_0 = \cdots = PGx_k = 0; \qquad k \geqq 0$$

and $PGx_i \neq 0$ for $k < i \leqq n-1$. Now, from all those systems, let us choose one for which k is maximal. If $k = n-1$, then $PG = 0$ which contradicts that $0 \neq G \geqq F$; thus $0 \leqq k < n-1$.

Remark that if $Pv_0 = 0$ for some $v_0 \in GX$ then, by repeating arguments already used in this proof we can show that

$$v_0 \in \bigvee_{i=0}^{k} \mathfrak{M}(Gx_i)$$

otherwise there exists $0 \neq G_1 \leqq G$ such that

$$G_1X = \mathfrak{M}(G_1v_0) \vee \bigvee_{i=0}^{n-2} \mathfrak{M}(G_1x_i)$$

and this fact contradicts the maximality of k. Thus $\bigvee_{i=0}^{k} \mathfrak{M}(Gx_i)$ is the null space of the restriction of P to GX.

Now, let us consider all the systems

$$\{0 \neq H \in \mathfrak{B}; H \leqq G; x_0, x_1, \cdots, x_k, \tilde{x}_{k+1}, \cdots, \tilde{x}_{n-1}\}$$

such that

$$HX = \bigvee_{i=0}^{k} \mathfrak{M}(Hx_i) \vee \bigvee_{i=k+1}^{n-1} \mathfrak{M}(H\tilde{x}_i)$$

and $(I-P)H\tilde{x}_{k+j} = 0$; $j = 1, \cdots, l$ while $(I-P)H\tilde{x}_{k+j} \neq 0$ for $j = l+1, \cdots, n-1-k$. From all these systems choose one for which l is maximal and assume that $(I-P)w_0 = 0$ for some $w_0 \in HX$. Then, by the same kind of argument one can easily see that

$$w_0 \in \bigvee_{i=0}^{k} \mathfrak{M}(Hx_i) \vee \bigvee_{i=k+1}^{k+l} \mathfrak{M}(H\tilde{x}_i)$$

and further

$$w_0 \in \bigvee_{i=k+1}^{k+l} \mathfrak{M}(H\tilde{x}_i)$$

i.e., the null space of $I - P/_{HX}$ coincides with $\bigvee_{i=k+1}^{k+l} \mathfrak{M}(H\tilde{x}_i)$. Hence HX can be decomposed as a direct sum as follows

$$HX = \bigvee_{i=0}^{k} \mathfrak{M}(Hx_i) \oplus \bigvee_{i=k+1}^{k+l} \mathfrak{M}(H\tilde{x}_i)$$

where $l \geqq 1$ since $k < n-1$ and H has uniform multiplicity n. This contradiction shows that $x_0 = 0$.

Finally, suppose that $P \neq F$. Then $Px - Fx \neq 0$ for some $x \in X$ and $P(Px - Fx) = 0$; so we get a contradiction to the first part of the proof. Thus $P = F \in \mathfrak{B}$. The converse is obvious. Q.E.D.

Theorem 3. *Let $\mathfrak{B}$ be of type (D). Every operator commuting with $\mathfrak{B}$ is spectral and its scalar part belongs to $\mathfrak{A}(\mathfrak{B})$.*

Proof. By Foguel [7, Theorem 2.3 and Lemma 2.2], for every operator $T \in \mathfrak{B}^c$ there corresponds a sequence of Borel sets $\{\alpha_m\}$ increasing to Ω and such that

$$TE(\alpha_m) = \sum_{i=1}^{n} S(f_i \chi_{\alpha_m}) P_{i,m} + N_m; \qquad m = 1, 2, \cdots$$

where f_i; $i = 1, 2, \cdots, n$ are bounded measurable functions, $|f_i(\omega)| \leqq \| T \|$ a.e.; $P_{1,m}, P_{2,m}, \cdots, P_{n,m}$ disjoint projections commuting with $\mathfrak{B}$ and N_m a nilpotent of order n. By lemma 2 $P_{i,m} \in \mathfrak{B}$; $i = 1, 2, \cdots, n$; $m = 1, 2, \cdots$, thus

$$TE(\alpha_m) = S(g_m) + N_m; \qquad m = 1,2,\cdots$$

where g_m; $m = 1,2,\cdots$ are bounded measurable functions and $|g_m(\omega)| \leqq \|T\|$ a.e. Then, for $k \leqq m$ we shall get

$$TE(\alpha_k) = S(g_m\chi_{\alpha_k}) + N_mE(\alpha_k)$$

hence

$$\begin{cases} S(g_m\chi_{\alpha_k}) = S(g_k) \\ N_mE(\alpha_k) = N_k \end{cases}$$

and further $g_m(\omega) = g_k(\omega)$ for almost every $\omega \in \alpha_k$. Denote

$$g(\omega) = g_k(\omega); \qquad \omega \in \alpha_k;\ k = 1,2,\cdots$$

Then g is a bounded measurable function and

$$\lim_{m\to\infty} g_m(\omega) = g(\omega)$$

for a.e. $\omega \in \Omega$. By [6 Theorem, IV–10–10]

$$\lim_{m\to\infty} S(g_m)x = S(g)x; \qquad x \in X$$

Now, if $N = T - S(g)$ then

$$Nx = \lim_{m\to\infty} N_mx; \qquad x \in X$$

and, consequently N will be a nilpotent belonging to $\mathfrak{B}^c$. In conclusion $T = S(g) + N$ where $S(g) \in \mathfrak{A}(\mathfrak{B})$ and $N^n = 0$. Q.E.D.

Corollary 4. *Let $\mathfrak{B}$ be of type (D) such that $\mathfrak{B}^c$ contains no nilpotent operator. Then $\mathfrak{B}^c = (\mathfrak{B}^c)^c = \mathfrak{A}(\mathfrak{B})$.*

It can be shown that there are no nilpotent operators commuting with the B.A. of projections constructed by Dieudonné [4]; thus both its commutants coincide with the algebra generated by the B.A. in the uniform operator topology.

Lemma 5. *For any B.A. of projections of finite uniform multiplicity $\mathfrak{B}$ there exists $0 \neq E_0 \in \mathfrak{B}$ such that*

$$E_0X = X_1 \oplus \cdots \oplus X_k$$

where X_j; $j = 1,2,\cdots,k$ are subspaces invariant under $\mathfrak{B}$; $\mathfrak{B}$ restricted to X_j has finite uniform multiplicity n_j, ($\sum_{j=1}^k n_j = n$) and is of type (D).

Proof. If $\mathfrak{B}$ is of type (D) the assertion is trivial. If it is not type (D), then we can find $0 \neq F \in \mathfrak{B}$ and $z_i \in X$; $i = 1,2,\cdots,n$ such that

$$FX = \left[\bigvee_{i=1}^{p} \mathfrak{M}(Fz_i)\right] \oplus \left[\bigvee_{i=p+1}^{n} \mathfrak{M}(Fz_i)\right]; \qquad 1 \leqq p \leqq n-p$$

Among all these systems $\{F \mid 0 \neq F \in \mathfrak{B};\ z_1, \cdots, z_n\}$ choose one for which p is minimal. Denote

$$X_1 = \bigvee_{i=1}^{p} \mathfrak{M}(Fz_i);\ Y_1 = \bigvee_{i=p+1}^{n} \mathfrak{M}(Fz_i)$$

Then X_1 and Y_1 are subspaces invariant under $\mathfrak{B}$; $\mathfrak{B}$ restricted to X_1 has finite uniform multiplicity p and $\mathfrak{B}$ restricted to Y_1 has finite uniform multiplicity $n-p$. Furthermore, in view of the minimality of p, $\mathfrak{B}$ restricted to X_1 is of type (D).

Repeating this process for Y_1 and so on, we shall finish the proof after a finite number of operations. Q.E.D.

LEMMA 6. *Assume there is no non-trivial nilpotent operator in $\mathfrak{B}^c$ and let $T \neq 0$ be an operator commuting with $\mathfrak{B}$. Then, there exists $E_0 \in \mathfrak{B}$ such that $0 \neq TE_0 \in (\mathfrak{B}^c)^c$.*

Proof. Denote

$$F_0 = \bigvee \{E \mid E \in \mathfrak{B};\quad TE = 0\}$$

$$F = I - F_0.$$

Then $F \in \mathfrak{B}$ and $TF = T \neq 0$. Using Lemma 5 for FX we shall get a projection $0 \neq E_0 \leqq F$; $E_0 \in \mathfrak{B}$ such that

$$E_0 X = X_1 \oplus X_2 \oplus \cdots \oplus X_k$$

and $\mathfrak{B}$ restricted to X_j; $j = 1, 2, \cdots, k$ is of type (D). Let P_j be the projection on X_j. Then

$$TE_0 = \sum_{j=1}^{k} P_j T$$

and if $j \neq h$ $P_j T P_h$ is evidently a nilpotent commuting with $\mathfrak{B}$; thus $P_j T P_h = 0$. But $P_j T P_j$ can be considered as an operator in X_j which commutes with $\mathfrak{B}_{/X_j}$. Hence by Corollary 4

$$P_j T E_0 = P_j T P_j = S(f_j) P_j; \qquad j = 1, 2, \cdots, k$$

where f_j is a bounded measurable function. Consequently

$$TE_0 = \sum_{j=1}^{k} S(f_j) P_j$$

and, further, for another operator A which commutes with $\mathfrak{B}$

$$AE_0 = \sum_{j=1}^{k} S(g_j) P_j .$$

Thus

$$(TE_0)A = (TE_0)(AE_0) = \sum_{j=1}^{k} S(f_j)S(g_j)P_j = (AE_0)(TE_0) = A(TE_0)$$

i.e. $TE_0 \in (\mathfrak{B}^c)^c$. Finally, let us remark that $TE_0 \neq 0$ since $0 \neq E_0 \leqq F$. Q.E.D.

THEOREM 7. *Assume there is no non-trivial nilpotent operator in $\mathfrak{B}^c$. Then $\mathfrak{B}^c = (\mathfrak{B}^c)^c$.*

Proof. Let $0 \neq T \in \mathfrak{B}^c$. Denote

$$\mathfrak{B}_0 = \{E(\delta) \,|\, 0 \neq TE(\delta) \in (\mathfrak{B}^c)^c\}$$

and remark that by previous lemma, $\mathfrak{B}_0$ is not void. If $\{E(\delta_\gamma)\}$ is an increasing chain in $\mathfrak{B}_0$ then by [1 Lemma 2.3]

$$\begin{aligned} A[T \bigvee_\gamma E(\delta_\gamma)]x &= AT \lim_\gamma E(\delta_\gamma)x = \lim_\gamma ATE(\delta_\gamma)x \\ &= \lim_\gamma TE(\delta_\gamma)Ax = [T \bigvee_\gamma E(\delta_\gamma)]Ax; \qquad x \in X,\ A \in \mathfrak{B}^c \end{aligned}$$

Thus $\bigvee E_\gamma(\delta_\gamma) \in \mathfrak{B}_0$ and lemma of Zorn insures the existence of a maximal element $E(\Omega_0)$ of $\mathfrak{B}_0$. If $TE(\Omega - \Omega_0) \neq 0$, then by Lemma 6 for the subspace $E(\Omega - \Omega_0)X$ one can find $E(\sigma_0) \in \mathfrak{B}$; $\sigma_0 \subset \Omega - \Omega_0$; $0 \neq TE(\sigma_0) \in (\mathfrak{B}^c)^c$. Consequently $E(\Omega_0) \bigvee E(\sigma_0)$ belongs to $\mathfrak{B}_0$ which contradicts the maximality of $E(\Omega_0)$. Therefore, $T = TE(\Omega_0) \in (\mathfrak{B}^c)^c$. Q.E.D.

COROLLARY 8. *Let $\mathfrak{E}$ be a complete countably decomposable B.A. of projections containing no projections of infinite uniform multiplicity and such that there are no non-trivial nilpotent operators in $\mathfrak{E}^c$. Then $\mathfrak{E}^c = (\mathfrak{E}^c)^c$.*

Proof. By [2 Theorem, 3.4] $I = \bigvee_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} E_n$ where $E_n \in \mathfrak{E}$ are disjoint projections such that if $E_n \neq 0$, it has uniform multiplicity. Hence, our statement follows from theorem 7. Q.E.D.

3. **Remarks.** a. Dieudonné [3] has constructed another example of a B.A. of projections $\mathfrak{F}$ of finite uniform multiplicity ($n = 2$) for which there are no non-trivial nilpotent operators in its first commutant and $\mathfrak{A}(\mathfrak{F})$ is a proper subalgebra of $\mathfrak{F}^c = (\mathfrak{F}^c)^c$. It shows that in Theorem 7 $\mathfrak{B}^c = (\mathfrak{B}^c)^c$ must not coincide with $\mathfrak{A}(\mathfrak{B})$.

b. In the decomposition of T given by Theorem 3 a nilpotent part was obtained. *We do not know if it really exists.* The single example of a B.A. of type (D) which has been constructed until now is that of Dieudonné [4] and one can easily show that there is no nilpotent commuting with Dieudonné's B. A. of projections.

If this is the general case and the commutant of a B.A. of projections of type (D) always coincides with the algebra $\mathfrak{A}(\mathfrak{B})$, then we are able to prove the converse of Corollary 8, i.e. commutative of the commutant implies the absence of non-trivial nilpotent operators in the first commutant.

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